Closing today: 3.4(1)(2)
Closing *Tues*: 10.2
Closing Fri: 3.5(1)(2)
Office Hours - 1:30-3:00 in COM B-006

10.2 Parametric Equations *(continued)*

Recall: Given $x = x(t), y = y(t)$, we find the slope of the tangent line using

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Entry Task: The motion of a particular pitched baseball is given by

$$x(t) = 142t$$

$$y(t) = -16t^2 + 4t + 5$$

Find the equation of the tangent line at $t = 1/2$.

Example: Old test question

Find all points on

$$x(t) = t^2 + t + 3$$

$$y(t) = t^3 - 2$$

when the tangent line has slope 1.

Speed: For a parametric equation, it is natural to ask what the "speedometer" speed is for the moving object.

"average speed from *t* to *t+h*" $= \dfrac{\text{change in distance}}{\text{change in time}}$

$$\approx \frac{\sqrt{\big(x(t+h)-x(t)\big)^2+\big(y(t+h)-y(t)\big)^2}}{h}$$

$$= \sqrt{\left(\frac{x(t+h)-x(t)}{h}\right)^2 + \left(\frac{y(t+h)-y(t)}{h}\right)^2}$$

"instantaneous speed at *t*" is the limit of the above expressions as $h \to 0$

$$= \sqrt{(x'(t))^2 + (y'(t))^2}$$

Thus,

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example: Again,

$$x(t) = 142t$$

$$y(t) = -16t^2 + 4t + 5$$

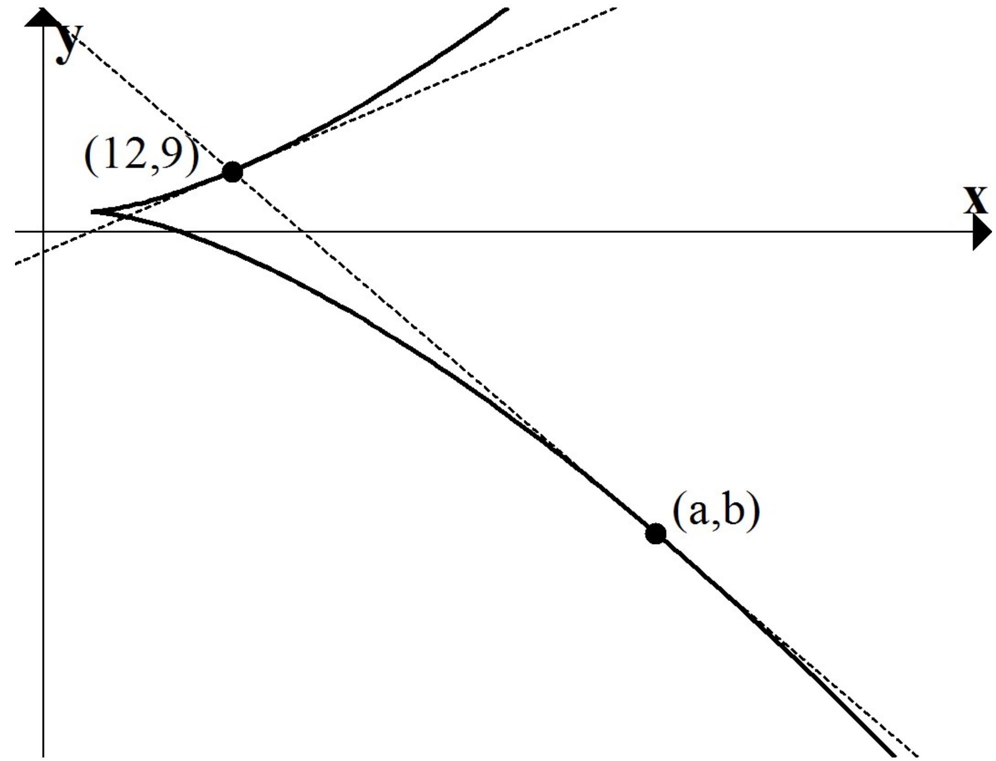
Find the speed of the ball at $t = 1/2$.

HW10.2 #7 Hint:

$x = 9t^2 + 3, y = 6t^3 + 3$

There are two tangent lines to this curve that **also** pass through (12,9).

Find these two tangent lines.



Special parametric equations:

1. Uniform Circular Motion:

$$\boldsymbol{x = x_c + r\cos(\theta_0 + \omega t)}$$
$$\boldsymbol{y = y_c + r\sin(\theta_0 + \omega t)}$$

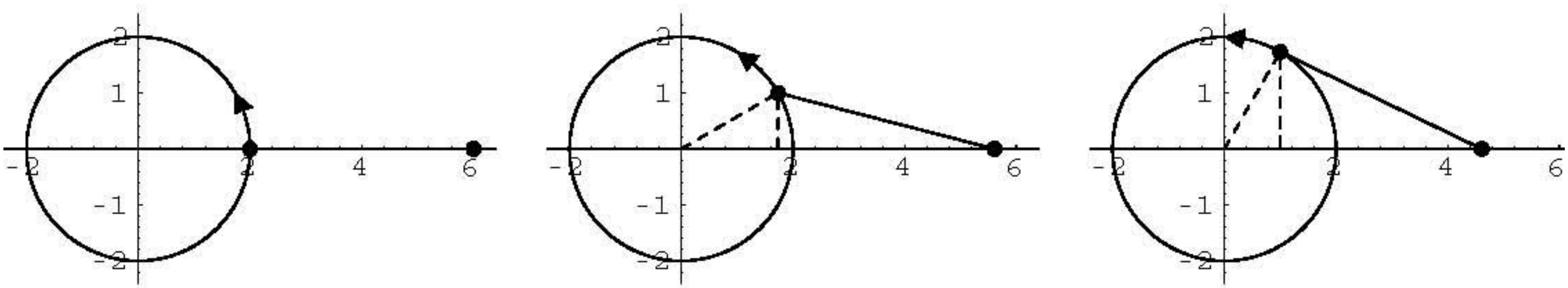
Note the fundamental circular motion facts from precalculus (*only* true in radians):

$$\text{linear speed} = v = \omega r,$$
$$\text{arc length} = s = r\theta$$

2. Uniform Linear Motion:

$$\boldsymbol{x = x_0 + at}$$
$$\boldsymbol{y = y_0 + bt}$$

From HW (Piston Problem): A 4cm rod is attached at one end to a point, A, on a wheel of radius 2 cm. The other end B is free to move back and forth along a horizontal bar that goes through the center of the wheel. At time *t=0* the rod is situated as in the diagram at the left below. The wheel rotates at 3.5 rev/sec.



Find parametric equation for the point A and the point B.

3.5 Implicit Differentiation

Motivation: Consider the unit circle

$$x^2 + y^2 = 1$$

Does NOT define a function. It *implicitly* defines more than one function.

$$y = f(x) = \sqrt{1 - x^2} \quad \text{or}$$

$$y = g(x) = -\sqrt{1 - x^2}$$

Questions:

1. Find $f'(x)$ and $g'(x)$.
2. What is the slope of the tangent line at $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$?

New idea (Implicit Differentiation):

Given $x^2 + y^2 = 1.$

Think of y as a function of x and differentiate directly to save time and energy (and gain simplicity).

So think of it as:

$$x^2 + (y(x))^2 = 1.$$

Again: What is the slope of the tangent line at $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$?

General Notes (Implicit Differentiation)

Given any equation of the form:

$$F(x, y) = 0,$$

we think of *y* as an *implicit* function of *x*

$$F(x, y(x)) = 0$$

and differentiate directly (correctly using the chain rule as we go!)

Quick Examples: Find dy/dx

1. $y^2 = x$

2. $x^2y + y^2 = 3$

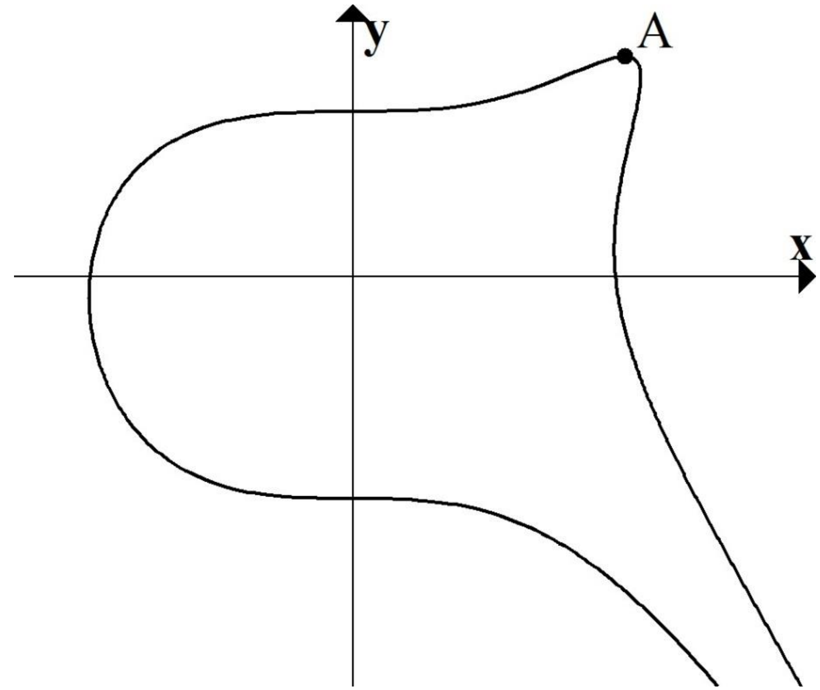
3. $xe^y + \tan(x) + \sin(y) = 1$

Old Midterm Question:

Consider the curve implicitly defined by

$$(x^3 - y^2)^2 + e^y = 4.$$

Find the (x, y) coordinates of the point *A* shown (highest point on the curve).



Inverse Functions:

We write inverse functions as

$y = f^{-1}(x)$ which is equivalent to

$f(y) = x$.

We can implicitly differentiate

$$\frac{d}{dx}[f(y) = x] \Rightarrow f'(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$$

Examples: Find dy/dx

1. $y = \sqrt{x}$

2. $y = \sin^{-1}(x)$

3. $y = \tan^{-1}(x)$

4. $y = \ln(x)$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\mathrm{x}\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{\mathrm{x}\sqrt{x^2-1}}$

- *Note*: The formulas all assume the principal domains as defined in our textbook.

Exercise: Find dy/dx

$$y = \tan^{-1}(e^{3x})$$

Now you can just use these shortcuts.